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Natural convection in an infinite porous medium with a concentrated heat source

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The phenomenon of buoyancy-induced convection in an infinite porous medium with a concentrated heat source is studied analytically. The transient and steady-state temperature distribution and flow pattern around the source are determined using a perturbation analysis in the Rayleigh number based on the heat generation rate at the source. The first-order transient solution derived in the paper is valid for Rayleigh numbers less than 10. The transient flow pattern consists of an expanding vortex ring situated in the horizontal plane containing the source. The steady-state solution, valid for Rayleigh numbers of the order of 20 or less, reveals an upward flow pattern which becomes very intense near the source. The upward flow extends throughout the medium. Both solutions show that as the Rayleigh number increases the region situated above the source is effectively heated by natural convection in addition to direct heat conduction from the source.

1. Introduction

Buoyancy-induced convection in fluid-saturated porous media has been the subject of many studies owing to its numerous and wide-ranging applications. For example, this class of phenomena is encountered in hydrology, petroleum geology, geophysics, thermal-insulation engineering and nuclear engineering. The object of this article is to analyse an important fundamental problem in porous-media free convection, namely the temperature field and ensuing flow pattern around a concentrated heat source suddenly embedded in an infinite fluid-saturated porous medium. In spite of its relevance to many practical situations, this phenomenon has not been analysed before. Most of the literature on buoyancy-induced convection in porous media, which, incidentally, is only thirty years old, deals primarily with phenomena where the heating or cooling is applied along the boundaries of the porous medium. For example, the Rayleigh–Bénard problem associated with heating a horizontal porous layer from below was studied by Horton & Rogers (1945), Lapwood (1948) and Wooding (1957). The end effects present when the horizontal layer is finite were investigated by Elder (1966*a*).

Considerable work has been done on free convection in enclosures filled with a porous medium where heating and cooling is applied along the lateral (vertical) boundaries. The interest in this topic is motivated in part by the growing emphasis on effective fibrous and granular insulation systems. The flow in vertical cavities was studied by, among others, Schneider (1963), Chan, Ivey & Barry (1970), Bankvall (1974) and Burns, Chow & Tien (1977). The natural convection in a horizontal porous medium subjected to an end-to-end temperature difference was investigated by Bejan & Tien (1978). The only work related to free convection caused by a point heat source in a saturated porous medium was done by Wooding (1963). Among a number of phenomena that he touched on was the steady-state high Rayleigh number behaviour of the flow around a point source. Relying on boundary-layer approximations analogous to the classical viscous theory, he showed that in the high Rayleigh number limit the flow and temperature fields at some distance above the source are described by a solution similar to the solution for a laminar round jet derived by Schlichting (1933). A similar approach was later used by Minkowycz & Cheng (1976) in the study of high Rayleigh number convection about a vertical cylinder embedded in a porous medium. In a general study on transient free convection in a porous medium, Elder (1966*b*) considered the evolution of a finite two-dimensional blob of hot fluid released from the solid base of a horizontal porous slab.

The present article considers the convection generated in the vicinity of a point heat source which is continuous in time and is suddenly embedded in an infinite porous medium. Two aspects of this problem are of interest: the transient, time-dependent temperature and flow pattern around the source and the steady regime attained as time approaches infinity. Unlike Wooding (1963), this study considers the low Rayleigh number behaviour of the ensuing flow. As emphasized later in § 5, the study of the low Rayleigh number behaviour is more appropriate in view of the limitations associated with using the Darcy flow model. The present problem is related somewhat to the one solved by Morton (1960) for weak thermal vortex rings created by the sudden release of a finite amount of heat at a point in a still viscous fluid. For the problem treated in this article, the fluid flow is governed by Darcy's law for flow through porous media and, since the point heat source is continuous in time, steady-state temperature and velocity fields are encountered.

As a summary of what is presented below, the mathematical problem is formulated in § 2. The transient flow and temperature patterns are derived analytically in § 3. A solution for the steady-state temperature and flow distributions around the source is presented in § 4. Finally, in § 5 we conclude the study by reviewing the asymptotic character of the solutions and the limitations of the Darcy flow model.

2. Problem statement

Consider the spherical polar co-ordinate system (r, θ, ϕ) shown in figure 1. A point source of strength q (watts) is situated at the origin with the $\theta = 0$ axis pointing vertically upwards. Since the problem is symmetric in the angular direction ϕ around the vertical axis, neither ϕ nor the ϕ velocity component w appears in the analysis. The fluid saturating the porous medium is Boussinesq-incompressible, its density ρ varying slightly as a result of temperature changes:

$$\rho = \rho_0[1-\beta(T-T_0)], \tag{1}$$

in which β is the volumetric coefficient of thermal expansion, T the temperature and the subscript 0 indicates the properties of a reference state.

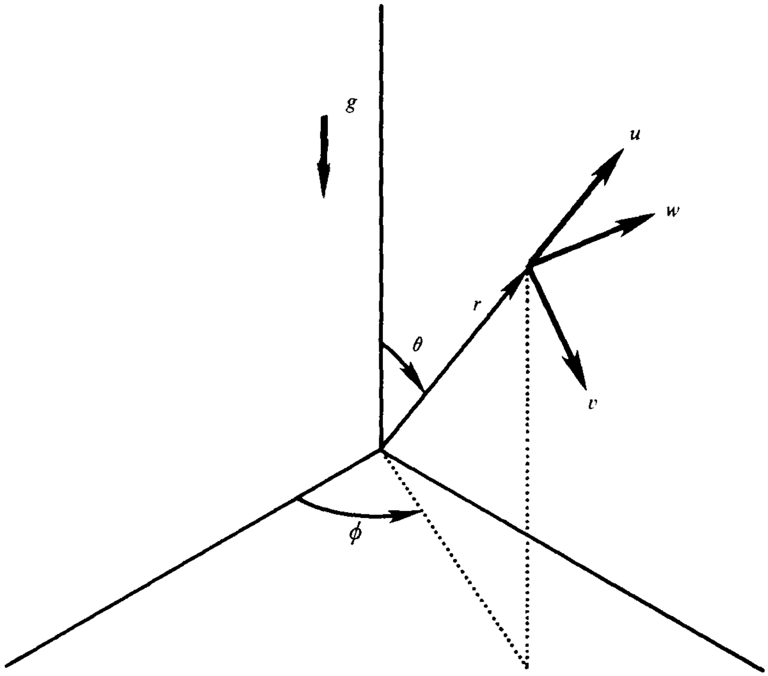

FIGURE 1. Spherical polar co-ordinate system.

The equations describing conservation of mass, momentum and energy in the medium are, respectively,

$$\frac{\partial}{\partial r}(r^2 u \sin\theta) + \frac{\partial}{\partial \theta}(rv\sin\theta) = 0, \tag{2}$$

$$u = -\frac{K}{\mu}\left(\frac{\partial P}{\partial r} + \rho g \cos\theta\right), \quad v = -\frac{K}{\mu}\left(\frac{1}{r}\frac{\partial P}{\partial \theta} - \rho g \sin\theta\right), \tag{3}$$

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right), \tag{4}$$

where u, v, t, P, g, K, μ and α stand for the radial and tangential velocity components, time, pressure, gravitational acceleration, medium permeability, viscosity and thermal diffusivity, respectively. In writing (2)–(4) we are modelling the porous medium as homogeneous. According to this commonly used model, the thermal diffusivity α of the medium is equal to $k/(\rho C_p)$, where k is the thermal conductivity of the saturated medium and ρC_p is the heat capacity of the fluid (see, for example, Elder 1966*a*). The momentum equations (3) are based on the assumption that Darcy's law applies; i.e. as summarized by Muskat (1946, p. 60), the Reynolds number based on the fluid velocity and average pore diameter is always less than one.

As usual, the governing equations are simplified if u and v are replaced by appropriately defining a stream function ψ which satisfies the continuity equations (2) identically:

$$u = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad v = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}. \tag{5}$$

Further, the pressure terms appearing in (3) are eliminated through cross-differentiation. The momentum and energy equations become

$$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2}=\frac{\beta g K}{\nu}\left[\cos\theta\frac{\partial T}{\partial\theta}+r\sin\theta\frac{\partial T}{\partial r}\right], \tag{6}$$

$$\frac{1}{\alpha}\left[\frac{\partial T}{\partial t}+\frac{1}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right)\right]=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right), \tag{7}$$

where ν is the kinematic viscosity μ/ρ.

Finally, (6) and (7) are put in a non-dimensional form by defining a new set of variables

$$\tau = t\alpha/K, \quad R = r/K^{\frac{1}{2}}, \quad \Theta = (T-T_0)kK^{\frac{1}{2}}/q, \quad \Psi = \psi/(\alpha K^{\frac{1}{2}}). \tag{8}$$

The resulting equations for the temperature Θ and the stream function Ψ are

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\Psi}{\partial R^2}=Ra\left(\cos\theta\frac{\partial\Theta}{\partial\theta}+R\sin\theta\frac{\partial\Theta}{\partial R}\right), \tag{9}$$

$$\frac{\partial\Theta}{\partial\tau}+\frac{1}{R^2\sin\theta}\left(\frac{\partial\Psi}{\partial\theta}\frac{\partial\Theta}{\partial R}-\frac{\partial\Psi}{\partial R}\frac{\partial\Theta}{\partial\theta}\right)=\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Theta}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right), \tag{10}$$

where Ra is a Rayleigh number based on the source strength q and the permeability K of the medium:

$$Ra = (\beta g/\alpha\nu)qK. \tag{11}$$

Equations (9) and (10) are to be solved subject to the initial conditions

$$u = 0, \quad v = 0, \quad T = T_0 \quad \text{at} \quad t = 0, \tag{12}$$

which are valid throughout the medium except at the origin. The boundary conditions are

$$u \to 0, \quad v \to 0, \quad T \to T_0 \quad \text{as} \quad r \to \infty, \tag{13}$$

$$\partial u/\partial\theta, v, \partial T/\partial\theta = 0 \quad \text{at} \quad \theta = 0, \pi. \tag{14}$$

Owing to the concentrated heat source, the origin is a singular point for both temperature and velocity. Consequently, u, v and T blow up as $1/r$ in the limit $r \to 0$. For the temperature, this behaviour is demonstrated by a heat balance over a spherical surface of radius zero containing the origin:

$$\lim_{r\to 0}\left[-k(4\pi r^2)\,\partial T/\partial r\right] = q. \tag{15}$$

3. The transient state

An analytical solution to the transient natural convection problem stated in the preceding section is possible in the limit of small Rayleigh numbers ($Ra \to 0$). The solution is found by means of a standard perturbation analysis which assumes power-series expressions in Ra for both Θ and Ψ:

$$\Theta = \Theta_0 + Ra\,\Theta_1 + Ra^2\Theta_2 + \ldots, \tag{16}$$

$$\Psi = \Psi_0 + Ra\,\Psi_1 + Ra^2\Psi_2 + \ldots. \tag{17}$$

The functions Θ_i and Ψ_i, with $i = 0, 1, 2 \ldots$, depend on τ, R and θ and are found by substituting (16) and (17) back into (9) and (10) and solving the equations obtained by collecting terms containing the same power of Ra. This procedure is straightforward, therefore only the final results are presented.

The zeroth-order functions Θ_0 and Ψ_0 correspond to the state of pure conduction around a continuous point source in a uniformly conducting medium. Since at $Ra = 0$ there is no fluid motion we can take $\Psi_0 = 0$. The transient temperature distribution in the medium is given by Carslaw & Jaeger (1959, p. 261) as

$$\Theta_0 = \frac{1}{4\pi R} \operatorname{erfc}\left(\frac{R}{2\tau^{\frac{1}{2}}}\right), \tag{18}$$

showing that for $\tau > 0$ the temperature at the origin is infinite, increasing as $1/R$ as R approaches zero.

The function Ψ_1 is obtained from (9) in combination with the functions Ψ_0 and Θ_0 derived already. Separation of variables is achieved by setting

$$\Psi_1 = (\tau^{\frac{1}{2}}/2\pi) \sin^2\theta f(\eta), \tag{19}$$

where $\eta = R/(2\tau^{\frac{1}{2}})$. The function $f(\eta)$ satisfies the ordinary differential equation

$$\eta^2 f'' - 2f = -(2/\pi^{\frac{1}{2}})\,\eta^2 \exp(-\eta^2) - \eta \operatorname{erfc}\eta. \tag{20}$$

Equation (20) has the general solution

$$f = \frac{c_1}{\eta} + c_2\eta^2 + \frac{\eta}{2}\operatorname{erfc}\eta + \frac{1}{4\eta}\operatorname{erf}\eta - \frac{1}{2\pi^{\frac{1}{2}}}\exp(-\eta^2), \tag{21}$$

where $$\operatorname{erf}\eta = \frac{2}{\pi^{\frac{1}{2}}}\int_0^\eta \exp(-x^2)\,dx, \quad \operatorname{erfc}\eta = 1 - \operatorname{erf}\eta.$$

Applying the $R\to\infty$ boundary condition (13), we find $c_2 = 0$. Since in the limit $R \to 0$ the velocities u and v increase as $1/R$, we find upon examining (5) that in this limit Ψ must be proportional to R (or η), hence $c_1 = 0$. We conclude that

$$\Psi_1 = \frac{1}{8\pi}\tau^{\frac{1}{2}}\sin^2\theta\left[2\eta\operatorname{erfc}\eta + \frac{1}{\eta}\operatorname{erf}\eta - \frac{2}{\pi^{\frac{1}{2}}}\exp(-\eta^2)\right]. \tag{22}$$

A set of streamlines corresponding to equal increments of $\Psi_1/\tau^{\frac{1}{2}}$ is shown in figure 2. The flow pattern at small Rayleigh numbers consists of a circular vortex whose radius increases in time as $\tau^{\frac{1}{2}}$. The centre of this vortex is located at $\eta = 0{\cdot}881$ in the horizontal plane containing the source. Near the origin the streamlines come close together, illustrating the fact that the velocity is infinite at the source. The same effect is shown in greater detail by the $Ra \to 0$ streamlines in figure 5, which depicts the steady-state ($\tau\to\infty$) flow pattern. As τ increases indefinitely, the flow pattern present near the origin spreads outwards, filling the entire space.

An equation for the first convective correction to the temperature field is provided by (10) in association with (18) and (22). Writing $\Theta_1 = \cos\theta F(\eta)/\tau^{\frac{1}{2}}$, (10) yields an ordinary differential equation for $F(\eta)$:

$$\eta^2F'' + 2(\eta^3+\eta)F' + 2(\eta^2-1)F$$
$$= -\frac{1}{64\pi^2}\left[\frac{1}{\eta^2}\operatorname{erfc}\eta + \frac{2}{\eta\pi^{\frac{1}{2}}}\exp(-\eta^2)\right]\left[2\eta\operatorname{erfc}\eta + \frac{1}{\eta}\operatorname{erf}\eta - \frac{2}{\pi^{\frac{1}{2}}}\exp(-\eta^2)\right]. \tag{23}$$

The general solution to this equation can be written as

$$F = \frac{1}{\eta^2}\exp(-\eta^2)\left(c_3 + c_4\int_0^\eta x^2\exp(x^2)\,dx\right) + F_p(\eta), \tag{24}$$

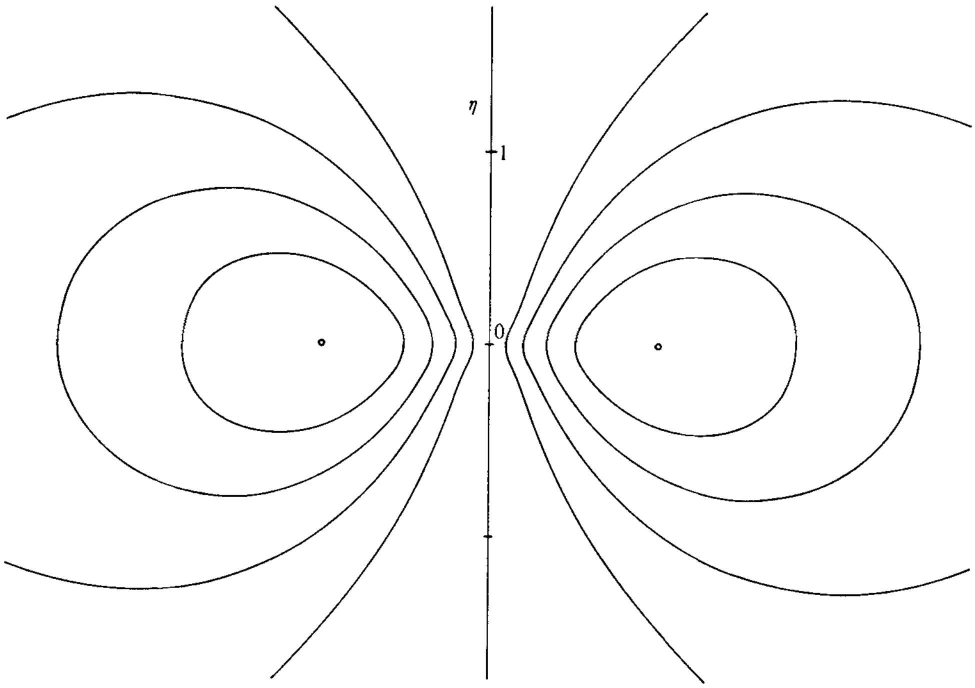

FIGURE 2. Transient natural convection pattern around point heat source; the lines correspond to equal increments of $\Psi_1/\tau^{\frac{1}{2}}$.

where $F_p(\eta)$ is the particular solution which satisfies (23). The first part of (24), containing the two constants c_3 and c_4, represents the general solution to the homogeneous form of (23); the homogeneous part of the solution was derived by using the general method of Frobenius. Applying the $\eta \to 0$ and $\eta \to \infty$ boundary conditions to expression (24), i.e. conditions (15) and (13), we conclude that the constants c_3 and c_4 must both be zero. Therefore the solution for Θ_1 is proportional to the particular solution $F_p(\eta)$, or $\Theta_1 = \cos\theta F_p(\eta)/\tau^{\frac{1}{2}}$. We were unable to guess a closed-form expression for $F_p(\eta)$ in the way in which we found expression (21) for f. Instead, we show here a series expansion for $F_p(\eta)$ valid in the important region near the source ($\eta \to 0$), i.e. the volume where the temperature field is affected most visibly by the source. Noting that F_p and Θ_1 behave as η^{-1} when η and R approach zero, we assume the series form

$$F_p = \sum_{n=-1}^{\infty} A_n \eta^n.$$

The coefficients A_n are easily identified by also expressing the right-hand side of (23) as a power series in $\eta < 1$ and collecting terms containing the same power of η. The final expression for Θ_1 is

$$\Theta_1 = \frac{\cos\theta}{64\pi^2\tau^{\frac{1}{2}}}\left(\frac{1}{\eta} - \frac{4}{3\pi^{\frac{1}{2}}} + \frac{6}{5\pi^{\frac{1}{2}}}\eta^2 - \frac{16}{45\pi}\eta^3 - \frac{152}{315\pi^{\frac{1}{2}}}\eta^4 + \frac{64}{315\pi}\eta^5 + \frac{517}{3780\pi^{\frac{1}{2}}}\eta^6 - \frac{992}{14175\pi}\eta^7 - \frac{2039}{69300\pi^{\frac{1}{2}}}\eta^8 + \frac{2591}{155929\pi}\eta^9 + \dots\right). \qquad (25)$$

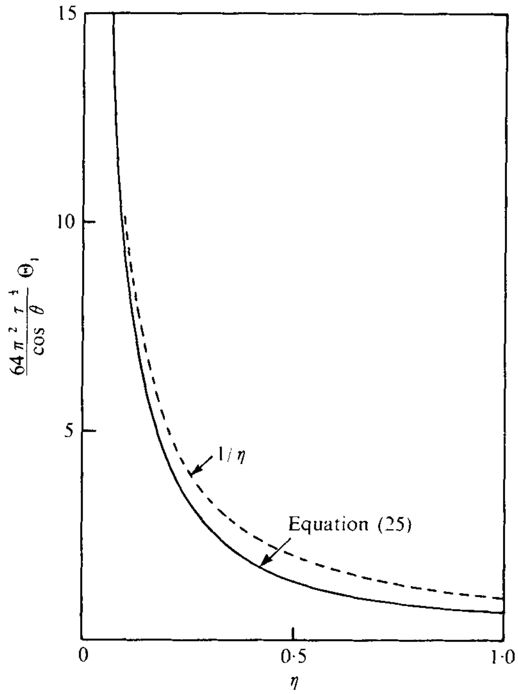

FIGURE 3. Radial dependence of the first-order convective correction to the transient temperature distribution.

The η dependence of the function Θ_1 is shown in figure 3. In the limit $\eta \rightarrow 0$ the first-order correction to the temperature field increases as $1/\eta$. Away from the source, Θ_1 drops more rapidly than $1/\eta$. This behaviour is to be expected since for $\eta \geqslant 1$ the presence of the source is felt only marginally. The power-series expression (25) is a very good description of Θ_1 in the spherical region $0 < \eta < 1$ surrounding the source. At $\eta = 1$, the η^9 term contributes only 0·82 % of the total estimate for Θ_1. As might have been expected, the $O(Ra)$ correction Θ_1 to the temperature field amounts to an increase in temperature throughout the upper half-space $(0 \leqslant \theta \leqslant \frac{1}{2}\pi)$ coupled with an equal and antipodal decrease in temperature in the lower half-space.

4. The steady state

As shown by the derivation of Θ_1 and Ψ_1 in the preceding section, the perturbation analysis of the transient problem is laborious and not well suited to be carried out beyond the first-order convective effect on the flow and temperature field. The analysis is somewhat simpler in the steady-state limit of this problem $(\tau \rightarrow \infty)$. In this limit the number of variables is reduced from three, τ, θ and η or τ, θ and R, to only two, θ and R.

It is known that in the steady state the temperature around a continuous point source embedded in an infinite conducting medium with no convection stabilizes to a spherically symmetrical distribution in which the temperature decreases as $1/R$ at points located further and further away from the source (see, for example, (18) in the limit $\tau \rightarrow \infty$). In this section we are concerned with the distortion imposed on the

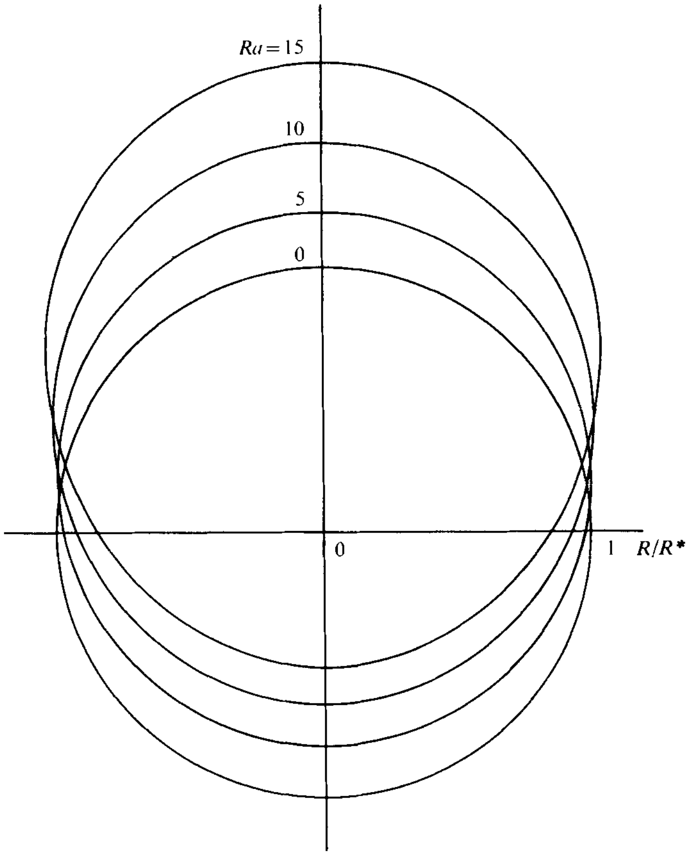

FIGURE 4. Steady-state temperature distribution; the lines represent the $(4\pi R^*)\,\Theta = 1$ isotherm for increasing values of Ra.

otherwise symmetrical distribution by the presence of convection heat transfer. We should expect that for a given Rayleigh number the region situated above the source will be warmer than the lower region since, in addition to direct conduction, it is also warmed by the upward flow engulfing the source.

The method for determining the functions Θ_i and Ψ_i making up the perturbation solution (16) and (17) is identical to the procedure described previously for the transient problem. For brevity, we show only the final expressions for Θ and Ψ with Ra terms up to the third power:

$$\Theta = \frac{1}{4\pi R}\left[1+\frac{1}{8\pi}\cos\theta\, Ra+\frac{5}{768\pi^2}\cos 2\theta\, Ra^2+\frac{1}{55296\pi^3}\cos\theta(47\cos^2\theta-30)\, Ra^3+\ldots\right], \tag{26}$$

$$\Psi = \frac{R}{8\pi}\left[\sin^2\theta\, Ra+\frac{1}{24\pi}\sin\theta\sin 2\theta\, Ra^2-\frac{5}{18432\pi^2}(8\cos^4\theta-3)\, Ra^3+\ldots\right]. \tag{27}$$

The temperature distribution (26) is illustrated in figure 4 by the temperature contour $\Theta = 1/4\pi R^*$, where R^* is a given radial distance. Increasing the Rayleigh number shifts the warm region upwards, although the temperature remains infinite at the

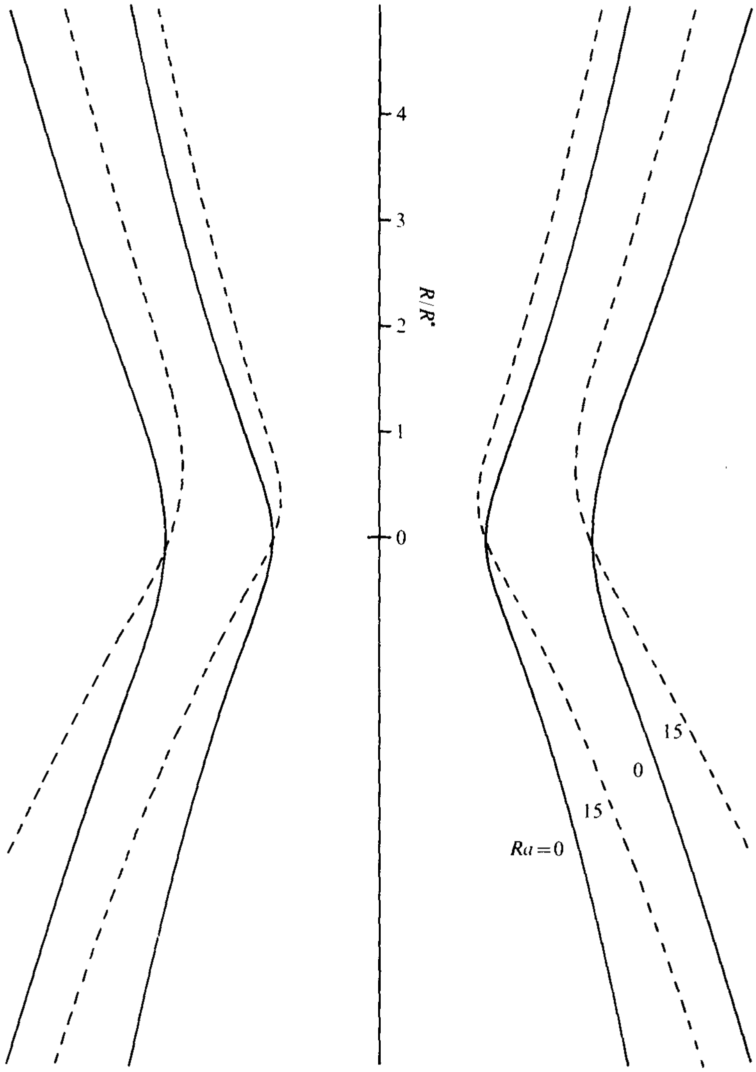

FIGURE 5. Steady-state flow pattern; the streamlines correspond to constant increments of $8\pi\Psi/(RaR^*)$.

origin. Figure 5 shows the steady-state flow pattern and its distortion as the Rayleigh number increases. The curves correspond to equal increments of the stream function $8\pi\Psi/(RaR^*)$, where R^* is again a fixed radial distance from the source. The streamlines show that as Ra increases the warmer region moves upwards. Consequently the fluid accelerates upwards in planes situated above the horizontal plane containing the source.

5. Concluding remarks

We have presented an asymptotic solution for the flow and temperature field around a concentrated heat source embedded in an infinite saturated porous medium. The solution was derived by performing a perturbation analysis in the Rayleigh number. The transient solution was carried out up to the first-order convective correction to the temperature and velocity field. As a result, it gives a reasonably accurate description as long as the Rayleigh number is of the order of 10 or less. The steady-state solution (26) and (27) was carried up to the third-order convective correction and is approximately valid for Rayleigh numbers of the order of 20 or less.

The problem analysed in this article models the situation arising when a small but finite volume inside a much larger fluid-saturated porous space is suddenly heated. The analysis was greatly simplified by considering the heat source to be concentrated at one point in an infinite medium. Consequently, the solutions for the temperature and flow field break down at the source, where the temperature and the fluid velocity blow up. In particular, the Darcy flow model ceases to be valid when the fluid velocity $U = (u^2+v^2)^{\frac{1}{2}}$ exceeds a critical value. Muskat (1946, p. 60) showed that when the Reynolds number based on U and the pore diameter D is approximately greater than one,

$$UD/\nu \gtrsim 1, \tag{28}$$

fluid inertial effects begin to be felt and equations (3) are no longer valid. However, in our case the region around the source in which the Darcy flow model breaks down ought to be small since the present study was carried out in the weak convection (small velocity) limit $Ra \to 0$. The limitation expressed by (28) is considerably more severe in the case of the $Ra \to \infty$ boundary-layer regime analysed by Wooding (1963) for a point heat source and Minkowycz & Cheng (1976) for a vertical heated cylinder embedded in a saturated porous medium.

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